The PyPWA Project
(A Partial-Wave/Amplitude Analysis Software Framework)
Version 0.1-alpha

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June 1, 2015

Abstract

The PyPWA Project aims to develop a software framework that can be used to perform Partial Wave and Amplitude Analysis (PWA) with the goal of extracting resonance information from multiparticle final states. More general, it can be used to fit a parametric model to the data by performing extended likehood fits. PyPWA is directed to photoproduction experiments using linear polarized photon beams. There are two basic software structures, one includes a general shell where any function (i.e. from an amplitude) can be used to fit and simulate data. A second approach starts with a framework that includes a realization of the Isobar model to calculate amplitudes, including extensions to Deck-type and baryon vertices corrections. Here we briefly describe the general formalism used in different sections of the code. Most of the code is written in Python, but hybrid code (in Cyhon or Fortran) has been used when appropiate. CERN-Root can also be used through PyRoot. The code makes full use of scripting and graphical interfaces (GUI) on the JLab Linux computer farm. Vectorization and parallel coprocessing (using Xeon-Phi or GPUs) will be also included in the near future. The goal of this software framework is to create a user friedly enviroment for analysis of linear polarized photoproduction experiments at Jefferson Lab.

Contents

1 Introduction

Our goal is the analysis of multiparticle final states produced by a beam of linearly polarized real photons (or quasi-real photons from electron inelastic scattering at very forward angles) striking a fixed target (we consider an unpolarized liquid Hydogen target). We aim at the identification of short lived (strongly interacting) resonances that have decayed to the observed multiparticle final states. The schematics of the considered reactions is shown in figure 3. The process of identification (finding a resonance and its quatum numbers) proceeds by fitting the data to theoretical models. The fiiting can be done through a Partial Wave descomposition or any Amplitude modeling of the characteristics of the data. This process normally entails to fit the overall yield and some angular distributions that are related by a parametric model. Different spin formalisms (helicity or covariant) account for spin-angular momentum couplings and incorporate QM conservations laws. Production mechanisms are generally unknow at these energies, they are fully or partially (using constraints from phenomenological theories) fitted to the data. The classic partial wave descomposition (truncated to low angular momenta contributions), represente a good first attemp to obtain a set of waves or amplitudes, *the model*, and then check that the data have been reasonable described by this *model*. This process has been used very successful when narrow and large resonances.

In this report we briefly describe the main components of the current PyPWA project.

1.1 General Shell for Fitting

1.1.1 Unbinned Extended Likelihood Fit

There are several ways to obtain the best parametric fit to a set of data, and several ways to evaluate their performance (goodness of the fit). We use the unbinned

extended likelihood method [**?**, **?**, **?**]. The vector $\overrightarrow{x}_i$ represents the set of variables necessary to define the particular configuration of an event i, and is of dimension n. We have measured N events in certain pahse-space, Δx^n, each given a set of measurements represented by a vector: $\overrightarrow{x}_i$, where i spans the set of events, i.e. $i = 1, ..., N$.

Our goal will be to find a mathematical parametrization (model) that explains these observations, i.e, that is able to explain (or predict) the number and "properties" of observed events. In general the model will be described by m parameters, $\{a_1, a_2, \cdots, a_m\} = \overrightarrow{a}$. We want to adjust the parameters in our model until we can best reproduce the observed data (fit). The probability of obtaining an event with the set $\overrightarrow{x}_i$ in our model is called $p(\overrightarrow{x}_i, \overrightarrow{a})$.

The standard likelihood of obtaining this arrangement for N measurements is the joint probability density

$$\mathscr{L} = \prod_{i=1}^{N} p(\overrightarrow{x}_i, \overrightarrow{a}) \tag{1}$$

with the normalization

$$\int_{\Delta x^n} p(\overrightarrow{x}, \overrightarrow{a}) d^n \overrightarrow{x} = 1 \tag{2}$$

.

We define a new extended likelihood that will also include the probability of observing N events by

$$\mathscr{L} = Prob(N) \prod_{i=1}^{N} p(\overrightarrow{x}_i, \overrightarrow{a}). \tag{3}$$

Assuming a Poisson distribution for the probability of observing N events, with an expected value of $\mathscr{N}$

$$Prob(N) = \frac{\mathscr{N}^N}{N!} e^{-\mathscr{N}} \tag{4}$$

the extended likelihood is then

$$\mathscr{L} = \left[\frac{\mathscr{N}^N}{N!} e^{-\mathscr{N}} \right] \prod_{i=1}^{N} p(\overrightarrow{x}_i, \overrightarrow{a}) \tag{5}$$

We renormalize $p(\overrightarrow{x}_i, \overrightarrow{a})$ by a new function $I(\overrightarrow{x}_i, \overrightarrow{a})$ such that

$$\eta(\overrightarrow{x}_i) I(\overrightarrow{x}_i, \overrightarrow{a}) = \mathscr{N} p(\overrightarrow{x}_i, \overrightarrow{a}) \tag{6}$$

The function $I(\overrightarrow{x}_i, \overrightarrow{a})$ is "nature's" production probability, it will be provided by the user. $\eta(\overrightarrow{x}_i)$ represents the detection acceptance (detector resolution is not yet taked into account in this version), therefore $\eta = 1$ if the an event is observed and $\eta = 0$ if the event is not observed. Values of η are also provided by the user after a Monte Carlo simulation of the detection process (i.e. Geant MC). Therefore

$$\int_{\Delta x^n} \eta(\overrightarrow{x}) I(\overrightarrow{x}, \overrightarrow{a}) d^n \overrightarrow{x} = \mathscr{N}. \tag{7}$$

For the observed data, therefore

$$I(\overrightarrow{x}_i, \overrightarrow{a}) = \mathscr{N} p(\overrightarrow{x}_i, \overrightarrow{a}) \tag{8}$$

The normalization $\mathscr{N}$ represents *the expected number of events to be observed in the full phase-space* of Δx^n. The value of $\mathscr{N}$ can be obtained from our model and knowing the measurement procedure (normally by Monte Carlo simulation, see below).

Then

$$\mathscr{L} = \left[\frac{\mathscr{N}^N}{N!} e^{-\mathscr{N}}\right] \prod_{i=1}^{N} \frac{I(\overrightarrow{x}_i, \overrightarrow{a})}{\mathscr{N}}. \tag{9}$$

Therefore

$$\mathscr{L} = \left[\frac{1}{N!} e^{-\mathscr{N}}\right] \prod_{i=1}^{N} I(\overrightarrow{x}_i, \overrightarrow{a}) \tag{10}$$

and taking the log on both sides

$$ln\mathscr{L} = -ln\left[N!\right] - \mathscr{N} + \sum_{i=1}^{N} ln\left[I(\overrightarrow{x}_i, \overrightarrow{a})\right]. \tag{11}$$

Then, substituting equation (7) and removing the constant term (unimportant in optimization)

$$ln\mathscr{L} = -\int_{\Omega} \eta(\overrightarrow{x}) I(\overrightarrow{x}, \overrightarrow{a}) d^n\overrightarrow{x} + \sum_{i=1}^{N} ln\left[I(\overrightarrow{x}_i, \overrightarrow{a})\right]. \tag{12}$$

and taking the negative values

$$-ln\mathscr{L} = -\sum_{i=1}^{N} ln\left[I(\overrightarrow{x}_i, \overrightarrow{a})\right] + \int_{\Omega} \eta(\overrightarrow{x}) I(\overrightarrow{x}, \overrightarrow{a}) d^n\overrightarrow{x} \tag{13}$$

Since the expected number of events is normally estimated by numerical Monte Carlo methods, we can use

$$\int_{\Omega} \eta(\overrightarrow{x}) I(\overrightarrow{x}, \overrightarrow{a}) d^n\overrightarrow{x} = \frac{1}{N_g} \sum_{i=1}^{N_g} \eta(\overrightarrow{x_i}) I(\overrightarrow{x}_i, \overrightarrow{a}) = \frac{1}{N_g} \sum_{i=1}^{N_a} I(\overrightarrow{x}_i, \overrightarrow{a}) \tag{14}$$

where N_g is the number of events generated and N_a is the number of events accepted in the Monte Carlo. We introduce the possibility of using a Q_i-factor (i.e., [**?**, **?**]), that for each data event, gives the probability for being a signal event (signal/(signal+background). We weight each log term on the data by this factor.

Therefore

$$\boxed{-ln\mathscr{L} = -\sum_{i=1}^{N} Q_i ln\left[I(\overrightarrow{x}_i, \overrightarrow{a})\right] + \frac{1}{N_g} \sum_{i=1}^{N_a} I(\overrightarrow{x}_i, \overrightarrow{a})} \tag{15}$$

We will find the best parameters $\overrightarrow{a}$ for our model minimizing the function $-ln\mathscr{L}$. The minimization of this function is performed by the python version (iMinuit) [**?**] of the CERN package MINUIT .

The user will privide the form of the function $I(\overrightarrow{x}_i, \overrightarrow{a})$, where $\overrightarrow{x}_i$ are the data variables (i.e. normally particles four-momenta or Mandelstham variables...) and $\overrightarrow{a}$ the set of parameters form the model to be fitted.

The errors in the parameters are given by the square root of their variances. It can be shown the the errors [**?**, **?**, **?**] are

$$\boxed{\sigma_{ij}^2 = E[(a_i - a_i^*)(a_j - a_j^*)] = E[(\overrightarrow{a} - \overrightarrow{a^*})^T(\overrightarrow{a} - \overrightarrow{a^*})] = [\mathscr{H}]_{ij}^{-1}.} \qquad (16)$$

The errors of the parameters can be calculated from the inverse of the Hessian matrix evaluated at the minimum. The MINUIT package may use different ways of calculating errors for more general cases (i.e. using the MINUIT package MINOS, see references [**?**], [**?**] and [**?**]).

1.2 General Shell for Simulation

A related problem to fitting is he simulation of a sample of events from a given a probability distribution, in our case, the function $I(\overrightarrow{x}_i, \overrightarrow{a})$. If we first fit the $\overrightarrow{a}$ parameters to the data, and then use $I(\overrightarrow{x}_i, \overrightarrow{a})$ to generated the sample, then this simulated sample can be used to check that the properties of the simulated data are modeling adequately the observed data. Therefore a goodness-of-fit check.

We can also use the simulation of a data sample using theoretically inspired parameters to compare with data or to check the efficiency of a giving fitting software. Our simulation is currently based in the Monte Carlo rejection methdd [**?**]. In the future, ee plan to study different algorithms for this purpose (Hastings-Metropolis,...).

We start by generating events from a "flat" phase-space distribution (i.e. all variables are equally probable). We then calculate for each generated event the value of $I(\overrightarrow{x}_i, \overrightarrow{a})$ and take the maximum value of this quantity in the full event set, I_{max}. We calculate the normalized probability for each event, $w_n = I(\overrightarrow{x}_i, w)/I_{max}$. For each event, we then generate a random number (RAN) between 0 and 1, if $w_n \geq RAN$ the event is kept in the sample, otherwise, the event is discarded. The final sample will then mirror a sample with a probability density function equal to $I(\overrightarrow{x}_i, \overrightarrow{a})$.

1.3 General Amplitude Formalism

For a general reaction amplitude $\mathscr{M}(\overrightarrow{x}, \overrightarrow{a})$, where $\overrightarrow{x}$ is a vector with the kinematical characteristics of each event (i), and $\overrightarrow{a}$ is a vector containing the parameters of the model, we can construct an intensity:

$$I(\overrightarrow{x}, \overrightarrow{a}) \equiv \sum_{ext.\ spins} |\mathscr{M}|^2 = \sum_{ext.\ spins} (\mathscr{M}\mathscr{M}^*) \qquad (17)$$

that is proportional to the probability of each event to be within the kinematical state, ($\overrightarrow{x}$), predicted by the model that is represented by the given amplitude. In general. we normally sum over the external particle spins (if they are not measured or determined).

As before, if the parameters of the amplitude (theory) are not known, they can be obtained by minimizing the negative form of the logarithmic unbinned extended likelihood function [1] written as

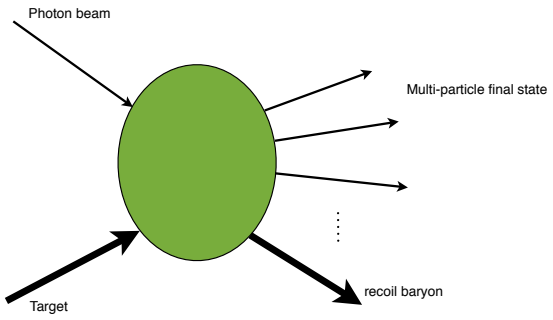


Figure 1: Photoproduction of a multi-particle final states.

$$-ln\mathscr{L} = -\sum_{i=1}^{N} ln\left[I(\overrightarrow{x}, \overrightarrow{a})\right] + \mathscr{N}. \tag{18}$$

where $\mathscr{N}$ represents the expected number of observed events. We then proceed in the same way as it was explained in the previous section. If we define a total fraction of events accepted, or total acceptance, as

$$\eta_x = \frac{N_a}{N_g} \tag{19}$$

Therefore, the function

$$\mathscr{F}(\overrightarrow{a}) = -ln\mathscr{L} = -\sum_{i=1}^{N} Q_i ln\left[\sum_{ext.\ spins}(\mathscr{M}\mathscr{M}^*)\right] + \eta_x \frac{1}{N_a}\sum_{i}^{N_a}\sum_{ext.\ spins}(\mathscr{M}\mathscr{M}^*). \tag{20}$$

is minimized to find the best values of $\overrightarrow{p}$. As before, Q_i is the probability that the event i is a signal. Many minimization (optimization) packages are available through SciPy (or NumPy). We use iMinuit as a default.

1.4 Partial Wave Analysis (PWA) in the Isobar Model

The Isobar Model fomalism is discussed in refence [1]. In the *isobar model*, a resonance is produced by the exchange of a meson or a Reggeon with the target and the

resonance will sequentially decay to the observed multiparticle final state. We describe the resonance decay via sequential 2-body decays. The considered meachnism is sketched in figure 2.

The reaction $\gamma N \to XN'$, will have two independent variables. We will take M, total mass of the final state set of particles, and t, Mandelstam t, to identify the kinematics. The differential cross section can be written, using Fermi's golden rule, by

$$\frac{d\sigma}{dtdM} \propto \sum_{ext.\ spins} \int |\mathscr{M}|^2 dx^n \tag{21}$$

and, if we consider small bins on M and t such that $\mathscr{M}$ only depends on $\overrightarrow{x}$, we can define

$$I(\overrightarrow{x}, \overrightarrow{V}) \equiv \sum_{ext.\ spins} |\mathscr{M}|^2 = \sum_{ext.\ spins} (\mathscr{M}\mathscr{M}^*) \tag{22}$$

Where (by tradition!) we have called our model parameters vector $\overrightarrow{V}$, where the componets are complex numbers (see below). $\mathscr{M}$ is a representation of the scattering operator or transition operator, $\widehat{T}$, given by

$$\mathscr{M} = \langle out|\widehat{T}|in\rangle \tag{23}$$

We have

$$I(\overrightarrow{x}, \overrightarrow{V}) \equiv \sum_{ext.\ spins} |\mathscr{M}|^2 = \sum_{ext.\ spins} \langle out|\widehat{T}|in\rangle(\langle out|\widehat{T}|in\rangle)^* \tag{24}$$

and, further

$$\langle out|\widehat{T}|in\rangle(\langle out|\widehat{T}|in\rangle)^* = \langle out|\widehat{T}|in\rangle\langle in|\widehat{T}^\dagger|out\rangle. \tag{25}$$

We define the operator $|in\rangle\langle in|$, corresponding to the initial state, the *initial spin density matrix operator*, $\widehat{\rho_{in}}$, as

$$\widehat{\rho_{in}} \equiv |in\rangle\langle in|. \tag{26}$$

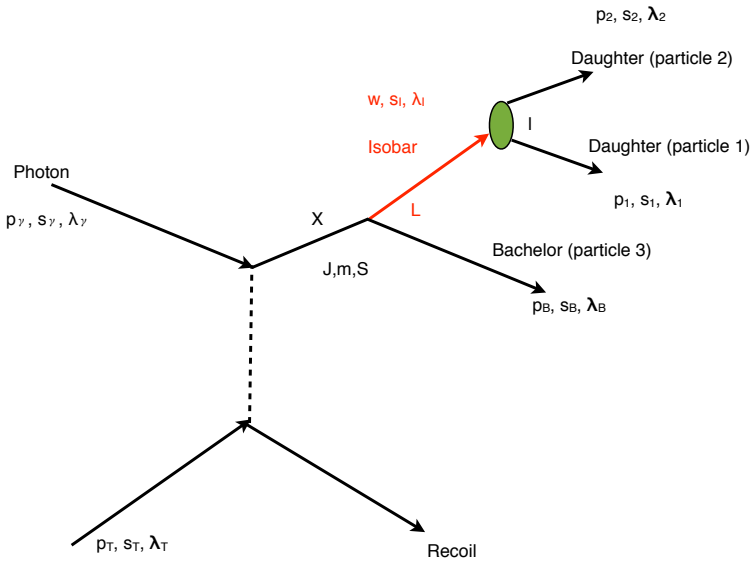
Suppose that we prepare the polarization of the incoming photons and target or measure their states of polarization. The average over their polarization will be completely described by this spin density matrix. In the case of a beam of polarized photons, any polarized state can be written as a linear combination of two pure polarization states. Therefore, the general structure of this 2×2 matrix (in any particular basis defined by $|i\rangle$ and $|j\rangle$) will be

$$\widehat{\rho_{in}} = \sum_{i,j} \rho_{i,j} = \sum_{i,j=1}^{2} |i\rangle\langle j|. \tag{27}$$

Then we have

$$I(\overrightarrow{x}, \overrightarrow{V}) = \sum_{ext.\ spins} \sum_{i,j} \langle out|^i\widehat{T}\rho_{ij}{}^j\widehat{T}^\dagger|out\rangle. \tag{28}$$

Here, in "*ext. spins*" we excluded the beam and target spins, as they are described by the initial state spin density matrix. The upper-left index on the transition operators



Where:
p_T,s_T $\boldsymbol{\lambda}_T$ four-momenta , spin and helicity of the target in the CM
$p_\gamma, s_\gamma, \lambda_\gamma$ four-momenta , spin and helicity of the photon (beam) in the CM

J, m, S are the resonance (X) total angular momentum, z-component and spin in GJ frame.
w is the mass of the isobar.
s_I and λ_I are the spin and helicity of the isobar in the GJ.
L the relative isobar-bachelor angular momentum in the GJ frame.
l the relative daughters angular momentum in the helicity frame.
$p_B, s_B, \boldsymbol{\lambda}_B$ four-momenta , spin and helicity of bachelor in the GJ frame.
$p_1, s_1, \boldsymbol{\lambda}_1$ four-momenta , spin and helicity of daughter 1 in the helicity frame.
$p_2, s_2, \boldsymbol{\lambda}_2$ four-momenta , spin and helicity of daughter 2 in the helicity frame.

Figure 2: Photoproduction and decay of a mesonic resonance in the Isobar Model.

correspond to the initial state specified by the spin density matrix. Keeping in mind the reaction represented in figure 3, we will assume that the transition operator can be factorized into two parts: the production (of X) and the decay operators (of X) such that:

$$I(\overrightarrow{x},\overrightarrow{V}) \propto \sum_{ext.\ spins} \sum_{i,j} \langle out|^i\widehat{T}_{decay}\ ^i\widehat{T}_{production}\rho_{ij}\ ^j\widehat{T}^{\dagger}_{production}\ ^j\widehat{T}^{\dagger}_{decay}|out\rangle \qquad (29)$$

Now we can take a complete orthogonal set of states, $|X\rangle$, such that $\sum_X |X\rangle\langle X| = 1$, and include them in the previous relation such that

$$I(\overrightarrow{x},\overrightarrow{V}) \propto \sum_{ext.\ spins} \sum_{i,j} \langle out|^i\widehat{T}_d \sum_X |X\rangle\langle X|^i\widehat{T}_p\rho_{ij}\ ^j\widehat{T}^{\dagger}_p \sum_{X'} |X'\rangle\langle X'|^jT^{\dagger}_d|out\rangle \qquad (30)$$

$$I(\overrightarrow{x},\overrightarrow{V}) \propto \sum_{ext.\ spins} \sum_{i,j} \sum_{X,X'} \langle out|^i\widehat{T}_d|X\rangle\langle X|^i\widehat{T}_p\rho_{ij}\ ^j\widehat{T}^{\dagger}_p|X'\rangle\langle X'|^j\widehat{T}^{\dagger}_d|out\rangle. \qquad (31)$$

The set of states , $|X\rangle$, are called *partial waves*, and gives the name of *partial wave analysis* (PWA) to the method presented here. Each of these states can be described by a set of quantum numbers that we will collectively call $\{b\}$. This set spans all the possible intermediate states, therefore, the experimental goal of finding the quantum numbers associated with the resonance is translated to measuring the partial wave amplitudes. We will call

$$\langle out|^i\widehat{T}_d|X\rangle \ = {}^iA_b(\overrightarrow{x}) \qquad (32)$$

the *decay amplitude* for a given wave, b, which may be calculated exclusively from the kinematical parameters ($\overrightarrow{x}$).

The production amplitude contains the hadronic QCD-based interaction that we are not able to calculate, rather the production amplitudes will be considered a *weight* on each partial decay amplitude of the final mix. These weights are the parameters to be fitted to the data, and will also depend on the k external spins. For example, in the case of an initial and final state nucleon (protons or neutrons), and no information about target (proton) spin, we will have $k = 2 \times 2 = 4$. We have assumed here that the resonance X decays to final spinless mesons. We will have

$$\langle X|^i\widehat{T}_p\rho_{ij}\ ^j\widehat{T}^{\dagger}_p|X'\rangle = {}^iV^k_b\rho_{ij}\ ^jV^{k*}_{b'} \qquad (33)$$

being V^k_b the production amplitudes. Note that the A's and V's are both complex numbers. Therefore

$$I(\overrightarrow{x},\overrightarrow{V}) = \sum_k \sum_{i,j} \sum_{b,b'} \ {}^iA_b(\overrightarrow{x})^iV^k_b\rho_{ij}\ ^jV^{k*}_{b'}\ {}^jA^*_{b'}(\overrightarrow{x}). \qquad (34)$$

The total number fo expected observed events is then

$$\mathscr{N} = \eta_x \frac{1}{N_a} \sum_i^{N_a} I(\overrightarrow{x}_i,\overrightarrow{V}) \qquad (35)$$

therefore

$$\mathscr{N} = \eta_x \frac{1}{N_a} \sum_{i}^{N_a} \sum_{k} \sum_{\epsilon,\epsilon'} \sum_{b,b'} {}^{\epsilon}A_b(\overrightarrow{x}_i)\ {}^{\epsilon}V_b^k\ \rho_{\epsilon,\epsilon'}{}^{\epsilon'}V_{b'}^{k*}\ {}^{\epsilon'}A_{b'}^*(\overrightarrow{x}_i). \tag{36}$$

If we assume that *all events are produced from the same vertex and by the same mechanism* (t-channel diffraction), the ${}^{\epsilon}V_b^k$ parameters are independent of the event number (i.e. they have the same structure for all events), the production parameters can be factored out of the event loop, giving

$$\mathscr{N} = \eta_x \sum_{k} \sum_{\epsilon,\epsilon'} \sum_{b,b'} {}^{\epsilon}V_b^k\ {}^{\epsilon'}V_{b'}^{k*} \frac{1}{N_a} \sum_{i}^{N_a} {}^{\epsilon}A_b(\overrightarrow{x}_i)\ \rho_{\epsilon,\epsilon'}{}^{\epsilon'}A_{b'}^*(\overrightarrow{x}_i). \tag{37}$$

Calling

$${}^{\epsilon,\epsilon'}\Psi_{b,b'}^x = \frac{1}{N_a} \sum_{i}^{N_a} {}^{\epsilon}A_b(\overrightarrow{x}_i)\ \rho_{\epsilon,\epsilon'}{}^{\epsilon'}A_{b'}^*(\overrightarrow{x}_i) \tag{38}$$

the *accepted normalization integral*, we obtain

$$\mathscr{N} = \eta_x \sum_{k} \sum_{\epsilon,\epsilon'} \sum_{b,b'} {}^{\epsilon}V_b^k\ {}^{\epsilon'}V_{b'}^{k*}\ {}^{\epsilon,\epsilon'}\Psi_{b,b'}^x. \tag{39}$$

Notice that this integral needs to be calculated only once during the minimization process, saving computer resources.

Including equations (**??**) and (39) into the likelihood function, equation (20), we have

$$\begin{aligned} -ln\mathscr{L} \propto -\sum_{i=1}^{N} ln\left[\textstyle\sum_k \sum_{\epsilon,\epsilon'} \sum_{b,b'} {}^{\epsilon}A_b(\overrightarrow{x}_i) {}^{\epsilon}V_b^k \rho_{\epsilon,\epsilon'}{}^{\epsilon'}V_{b'}^{k*}\ {}^{\epsilon'}A_{b'}^*(\overrightarrow{x}_i)\right] \\ +\eta_x \sum_{k} \sum_{\epsilon,\epsilon'} \sum_{b,b'} {}^{\epsilon}V_b^k\ {}^{\epsilon'}V_{b'}^{k*}\ {}^{\epsilon,\epsilon'}\Psi_{b,b'}^x. \end{aligned} \tag{40}$$

We can shorten our notation if we include in "a wave" all the quantun number as $\alpha = [\epsilon, b]$, and consider only rank one processes, then

$$\begin{aligned} -ln\mathscr{L} = -\sum_{i=1}^{N} ln\left[\textstyle\sum_{\alpha,\alpha'} A_\alpha(\overrightarrow{x}_i) V_\alpha \rho_{\epsilon,\epsilon'} V_{\alpha'}^*\ A_{\alpha'}^*(\overrightarrow{x}_i)\right] \\ +\eta_x \sum_{\alpha,\alpha'} V_\alpha\ V_{\alpha'}^*\ \Psi_{\alpha,\alpha'}^x. \end{aligned} \tag{41}$$

This function will be minimized to obtain the V_α values. To find the *true* or predicted number of events in the $\Delta M \Delta t$ bin, which we will call N_{true}, we take

$$N_{true} = \frac{1}{N_g} \sum_{i}^{N_g} I(\overrightarrow{x}_i, \overrightarrow{V}) \tag{42}$$

Then

$$N_{true} = \frac{1}{N_g} \sum_{i}^{N_g} \sum_{k} \sum_{\alpha,\alpha'} A_\alpha(\overrightarrow{x}_i)\ V_\alpha \rho_{\epsilon,\epsilon'} V_{\alpha'}^*\ A_{\alpha'}^*(\overrightarrow{x}_i) \tag{43}$$

and calling

$$\Psi^r_{\alpha,\alpha'} = \frac{1}{N_g}\sum_i^{N_g} A_\alpha(\overrightarrow{x}_i)\rho_{\epsilon,\epsilon'}A^*_{\alpha'}(\overrightarrow{x}_i) \tag{44}$$

the *raw* normalization integral. Then

$$N_{true} = \sum_{\alpha,\alpha'} V_\alpha V^*_{\alpha'}\Psi^r_{\alpha,\alpha'} \tag{45}$$

and the yield for each partial wave is

$$N_{\alpha,true} = V_\alpha V^*_\alpha \Psi^r_{\alpha,\alpha} = \left|V_\alpha\right|^2 \Psi^r_{\alpha,\alpha}. \tag{46}$$

1.5 Extensions of the Isobar Model

If the model includes amplitudes related to other vertices or non-t-channel production mechanism (for example the Deck effect, Baryon contaminations, etc.) the factorization used in (39) is not always possible, and the accepted and raw normalization integrals will not factor. This has a very important effect in the time expended in the minimization process, as the normalization integrals need to be recomputed at each minimization step. THESE OPTIONS ARE NOT YET FULLY IMPLEMENTED IN THIS RELEASE.

After we obtain the ${}^\epsilon V^k_b$ values, we are able to generate MC events through our partial wave model and *predicted* many distributions of data properties (i.e., angular distributions, t-distributions, etc.) to compare directly with data. These comparisons allow the verification of the fit (see section **??**). To make the predictions we use the values of $I(\tau; {}^\epsilon V^k_b)$ to weight a generated (raw), phase-space, sample and then apply a detector simulation to produce a sample of observed events.

1.6 Simulation of Resonances and Waves

The simulation software can be use to simulate any combinition of resonances and waves. We want to generate simulated data from a set of $\alpha = 1, \cdots, N$ waves coming from $R = 1, \cdots, n$ resonances.

Let's assume a resonance, R, with a relativistic Breit-Wigner mass distribution (we might, of course, assume any other distribution) such that

$$\frac{d\sigma}{dw} = \frac{1}{(w_o^2 - w^2)^2 + w_o^2\Gamma^2} w_o^2\Gamma_o^2 F_l^2(q). \tag{47}$$

Let's take $\Gamma = \Gamma_o$ and the Blatt-Weisskopf coefficients $F_l^2(q) = 1$. The number of events for a given mass, w, are then

$$N_R(w) = C_R \frac{1}{(w_o^2 - w^2)^2 + w_o^2\Gamma_o^2} w_o^2\Gamma_o^2. \tag{48}$$

where C_R is a normalization coefficient that correspond to the number of events expected at $w = w_o$ (maximum). The value of these coefficients needs to be assumed to obtain the desired relation of cross sections among the resonances. A wave α will contribute to this number of events with a weight $W_{R,\alpha}$, that will also need to be assumed, such that

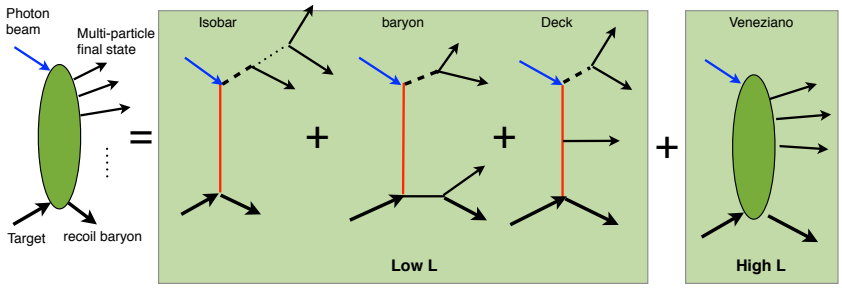


Figure 3: Photoproduction of a multi-particle final states. Extension of Isobar model.

$$N_{R,\alpha}(w) = W_{R,\alpha} C_R \frac{1}{(w_o^2 - w^2)^2 + w_o^2 \Gamma_o^2} w_o^2 \Gamma_o^2. \tag{49}$$

By equation (46), this number of events is also given by

$$N_{R,\alpha}(w) = \left|V_{R,\alpha}(w)\right|^2 \Psi^r_{\alpha,\alpha}(w) \tag{50}$$

therefore, from equations (49) and (50) we have

$$\left|V_{R,\alpha}(w)\right| = \sqrt{\frac{1}{\Psi^r_{\alpha,\alpha}(w)} W_{R,\alpha} C_R \frac{1}{(w_o^2 - w^2)^2 + w_o^2 \Gamma_o^2} w_o^2 \Gamma_o^2} \tag{51}$$

the magnitude of the contribution of the wave α to the resonance R at the value of the mass w. Values for the magnitude and angle of polarization ($\mathscr{P}$, α) will also need to be assumed to calculate the photon spin density matrix, $\rho_{\epsilon,\epsilon'}$, and the normalization integrals, $\Psi^r_{\alpha,\alpha}(w)$.

Recalling equation (86), the amplitude given by the Breit-Wigner formula is

$$V_{R,\alpha}(w) = \frac{\sqrt{C_R W_{R,\alpha}} w_o \Gamma_o}{(w_o^2 - w^2 - i w_o \Gamma_o(w))} = \left|V_{R,\alpha}(w)\right| e^{i\Phi_{R,\alpha}(w)} \tag{52}$$

where $\Phi_{R,\alpha}$ is the BW phase of the contribution of the wave α to the resonance R. Therefore

$$\Phi_{R,\alpha}(w) = \arctan \frac{w_o^2 \Gamma_o}{(w_o^2 - w^2)} \tag{53}$$

from which we can obtain $\Phi_{R,\alpha}$ and, therefore, the complex structure of the $V_{R,\alpha}(w)$ coefficients. The simulated mass spectrum is

$$N(w) = \sum_{R,R'}^{n} \sum_{\alpha,\alpha'}^{N} V_{R,\alpha}(w) V^*_{R',\alpha'}(w) \Psi^r_{\alpha,\alpha'}(w) \tag{54}$$

where we include all quantum numbers in one index defining a wave, $\alpha = (b, \epsilon, k)$. Notice that this calculation accounts for the interference between waves.

Suppose that we want to simulate a set of events produced by the previous set of waves and resonances. The probability that an event with characteristics given by $\overrightarrow{x}_i$ is produced, in a given mass bin w, is given by

$$I(\overrightarrow{x}_i, w) = \sum_{R,R'} \sum_{\alpha,\alpha'} V_{R,\alpha}(w) A_\alpha(\overrightarrow{x}_i) \rho_{\epsilon,\epsilon'} V^*_{R',\alpha'}(w) A^*_{\alpha'}(\overrightarrow{x}_i). \tag{55}$$

As it was explained before, we start by generating events from a phase-space distribution. We then calculate for each generated event the value of $I(\overrightarrow{x}_i, w)$ using (55) and take the maximum value of this quantity in the full event set, I_{max}. We calculate the normalized probability for each event, $w_n = I(\overrightarrow{x}_i, w)/I_{max}$. For each event, we then generate a random number (RAN) between 0 and 1, if $w_n \geq RAN$ the event is kept in the sample, otherwise, the event is discarded. The final sample will then mirror a sample produced by the waves and resonances assumed. When the V values used in (55) are the fitted values, this is the method used to generate a predicted sample of events.

1.7 Phase motion

From the mass-independent fit we obtained complex amplitudes for each wave in each mass bin. We have used their magnitudes (intensities) to calculate predicted counts and distributions, but we can also use their complex phase to look for resonant behavior. A single wave phase is arbitrary and thus ambiguous, therefore, only the difference of phases between two waves contains physical information. We normally examine the behavior of the phase difference (phase motion) of the wave under study against a well established resonant wave.

We can extract (from data) the phase differences between two waves and their errors. Equation (46) gives the number of predicted events in each mass bin for a given wave, we can write

$$N_\alpha = \left|V_\alpha(\Psi^r_{\alpha,\alpha})^{1/2}\right|^2 \tag{56}$$

therefore, the predicted number of events can be considered as the square of a complex amplitude $\mathcal{A}_b$ such as

$$\mathcal{A}_b = V_\alpha(\Psi^r_{\alpha,\alpha})^{1/2}. \tag{57}$$

The phase difference between two waves is then obtained from

$$\Delta\phi = (\phi_1 - \phi_2). \tag{58}$$

We will write $\mathcal{A}_1 = A_1 e^{i(\phi_1+\delta)}$ and $\mathcal{A}_2 = A_2 e^{i\phi_2}$, with an arbitrary phase (δ) since only the magnitude is derived from data. Therefore (with $C = e^\delta$ a constant),

$$\mathcal{A}_1\mathcal{A}_2^* = CA_1A_2e^{i(\phi_1-\phi_2)} = CA_1A_2e^{i\Delta\phi} \tag{59}$$

since

$$\mathcal{A}_1\mathcal{A}_2^* = V_1V_2^*(\Psi^r_{1,1}\Psi^r_{2,2})^{1/2} \tag{60}$$

we have

$$CA_1A_2e^{i\Delta\phi} = V_1V_2^*(\Psi^r_{1,1}\Psi^r_{2,2})^{1/2} \tag{61}$$

and therefore

$$\boxed{\Delta\phi = arctan\left(\frac{Im(V_1V_2^*)}{Re(V_1V_2^*)}\right).} \tag{62}$$

A similar equation to (**??**) but using the fitted production amplitudes from the mass-independent analysis.

The errors in the phase differences are calculated by (error propagation) taken $\Delta\phi$ as a function of the V_1 and V_2^* values. Calling $a = Re(V_1V_2^*)$ and $b = Im(V_1V_2^*)$ we have

$$\sigma^2_{\Delta\phi} = \left(\frac{\partial\Delta\phi}{\partial a}\right)^2\sigma_a^2 + \left(\frac{\partial\Delta\phi}{\partial b}\right)^2\sigma_b^2 + 2\left(\frac{\partial\Delta\phi}{\partial a}\right)\left(\frac{\partial\Delta\phi}{\partial b}\right)\sigma^2_{ab} \tag{63}$$

where

$$\frac{\partial \Delta\phi}{\partial a} = \frac{a}{a^2+b^2} \; ; \; \frac{\partial \Delta\phi}{\partial b} = \frac{-b}{a^2+b^2}. \tag{64}$$

The errors in a and b are derived from the error matrix, $\mathscr{C}$, of equation (**??**). We have

$$\sigma^2_{a(b)} = \mathscr{J}_{a(b)} \cdot \mathscr{C} \cdot \mathscr{J}^T_{a(b)} \tag{65}$$

and

$$\sigma^2_{ab} = \mathscr{J}_a \cdot \mathscr{C} \cdot \mathscr{J}^T_b. \tag{66}$$

We use the notation $V_1 = v_{1R} + iv_{1I}$ and $V_2^* = v_{2R} - iv_{2I}$, therefore

$$V_1 V_2^* = (v_{1R}v_{2R} + v_{1I}v_{2I}) + i(v_{1I}v_{2R} - v_{1R}v_{2I}) \tag{67}$$

from which we found

$$\mathscr{J}_a = (v_{2R}, v_{2I}, v_{1R}, v_{1I}) \; and \; \mathscr{J}_b = (-v_{2I}, v_{2R}, v_{1I}, -v_{1R}). \tag{68}$$

1.8 Moments in 2-body decays

We have seen that, after we take care of the dynamic quantities by binning the data in (M,t) bins, the intensity, $\mathrm{I}(\overrightarrow{x}_i)$, depends mainly on angular quantities. This is exactly true in the case of two particles final states. Therefore, it is natural to try a multipole expansion of the intensity distribution [**?**, **?**].

We can see [**?**] that

$$H(LM) = \sum_{l.m,l',m'} (\frac{2l'+1}{2l+1})^{1/2} \; \rho_{l,m,l',m'} (l'm'LM|lm)(l'0L0|l0) \tag{69}$$

an equation providing a relation between the *partial waves* and the *moments*, where $\rho_{l,m,l',m'}$ is the spin density matrix. The other parentheses are Clebsh-Gordan coefficients. Therefore, the moments include, although somehow indirectly, information about the spin density matrix. From the definition

$$H(LM) = \int dx^n I(\overrightarrow{x}_i) D^L_{M0}(\phi, \theta, 0) \tag{70}$$

If we assume that $I(\overrightarrow{x}_i)$ is mostly uniform in the (M,t) bin under study, we can write

$$H(LM) = I(\overrightarrow{x}_i) \sum_i^N D^L_{M0}(\phi_i, \theta_i, 0) \tag{71}$$

We can obtain, then, directly from data the (unnormalized) moments

$$H(LM) = \sum_i^N D^L_{M0}(\phi_i, \theta_i, 0) \tag{72}$$

These are sums of Wigner-D function directly calculable from data, completely model independent. Including the values of the Wigner-D functions, the first few

moments (for two spinless particles final states) have the following forms (all angles are calculated in the Gottfried-Jackson frame):

$$H(00) = 1 \tag{73}$$

$$H(10) = cos(\theta) \tag{74}$$

$$H(11) = \frac{-1}{\sqrt{2}} sin(\theta)cos(\phi) \tag{75}$$

$$H(20) = \frac{1}{2} * (3cos^2(\theta) - 1) \tag{76}$$

$$H(21) = \frac{-\sqrt{3}}{2} sin(\theta)cos(\theta)cos(\phi) \tag{77}$$

$$H(22) = \frac{\sqrt{6}}{4}(1 - cos^2(\theta))cos(2\phi) \tag{78}$$

$$H(30) = \frac{1}{2}(5cos^3(\theta) - 3cos(\theta)) \tag{79}$$

$$H(31) = \frac{-\sqrt{3}}{4} sin(\theta)(5cos^2(\theta) - 1)cos(\phi) \tag{80}$$

$$H(32) = \sqrt{\frac{15}{8}}(1 - cos^2(\theta))cos(\theta)cos(2\phi) \tag{81}$$

$$H(33) = \frac{-\sqrt{5}}{4}(1 - cos^2(\theta))^{\frac{3}{2}} cos(3\phi) \tag{82}$$

$$H(40) = \frac{1}{8}(35cos^4(\theta) - 30cos^2(\theta) + 3) \tag{83}$$

$$H(41) = \frac{-\sqrt{5}}{4} sin(\theta)(7cos^3(\theta) - 3cos(\theta))cos(\phi) \tag{84}$$

$$H(42) = \sqrt{\frac{5}{32}}(1 - cos^2(\theta))(7cos^2(\theta) - 1)cos(2\phi). \tag{85}$$

If we now weight our mass histogram (taken the number of events for each bin as a measure of $I(\overrightarrow{x}_i)$) by these quantities and plot versus mass, the first moment (H(00)) will give us the mass distribution and the others will give us information on the spin density matrix versus mass. We have code to calculate those moments.

1.9 Mass dependent fits

Fits to the mass distribution can be made using the general shell or a binned likelihood fit. In the isobar model we use the Breit-Wigner mas distribution as describe in reference [**?**]. After performing mass-independent fits in each bin of M (or M and t) we obtained the predicted distribution of $N_{true}(M)$ for each partial wave.

We use the relativistic Breit-Wigner (BW) prescription, with corrections, as explained below. The BW distribution represents just an approximation for the mass distribution ([**?**]). Future developments for high statistics experiments covering broader and less defined resonances will may use a more sophisticated parametrization as theory dictates.

As a default in a fitting we use

$$\Psi(w) = \frac{w_o \Gamma_o}{(w_o^2 - w^2 - i w_o \Gamma(w))}. \tag{86}$$

This is the well known relativistic Breit-Wigner (BW) mass distribution amplitude [**?**]. In the case where $\Gamma_o << w_o$, therefore $w \sim w_o$ and $\Gamma \sim \Gamma_o$; we obtain the non-relativistic case

$$\Psi(w) = \frac{\Gamma_o}{\frac{(w_o - w)(w_o + w)}{w_o} - i\Gamma_o} \sim \frac{\Gamma_o}{\frac{(w_o - w)(2w_o)}{w_o} - i\Gamma_o} \sim \frac{\Gamma_o/2}{w_o - w - i\Gamma_o/2}. \tag{87}$$

The measured (cross section) distribution is

$$\frac{d\sigma}{dw} = \left|V_l(w)\right|^2 pq \tag{88}$$

therefore

$$\frac{d\sigma}{dw} = e^{2i\delta(w)} \frac{\Gamma_o^2}{\Gamma(w)^2} sin^2\delta F_l^2(q) \tag{89}$$

or using

$$e^{2i\delta} = \frac{w_o^4}{(w_o^2 - w^2)^2 + w_o^2\Gamma^2} \tag{90}$$

and

$$sin^2\delta = \frac{\Gamma^2}{w_0^2} \tag{91}$$

we obtain

$$\boxed{\frac{d\sigma}{dw} = \frac{1}{(w_o^2 - w^2)^2 + w_o^2\Gamma^2} w_o^2 \Gamma_o^2 F_l^2(q).} \tag{92}$$

The F's are called the Blatt-Weisskopf centrifugal-barrier factors [**?**]. The first four are given by

$$F_0(q) = 1 \tag{93}$$

$$F_1(q) = \sqrt{\frac{2z}{z+1}} \tag{94}$$

$$F_2(q) = \sqrt{\frac{13z^2}{(z-3)^2 + 9z}} \tag{95}$$

$$F_3(q) = \sqrt{\frac{277z^3}{z(z-15)^2 + 9(2z-5)^2}} \tag{96}$$

with

$$z = (q/0.1973)^2 \; in \; GeV \tag{97}$$

The value of $q_R = 0.1973 GeV/c$ corresponds to a centrifugal barrier at 1 fermi.

The mass dependent amplitude must also be modified to account for the centrifugal barrier, and the new amplitude can be written as

$$V_l(w) \equiv e^{i\phi_l}\Psi(w)F_l(q) \tag{98}$$

where we have introduced a ϕ_l production phase for the wave l (which is independent of the mass).

$$\boxed{V_l(w) = e^{i(\phi_l+\delta(w))}\frac{\Gamma_o}{\Gamma(w)}sin\delta F_l(q)} \tag{99}$$

The observed mass distribution can be hard to fit through this function. Sometimes we can find a better fit to the data just phenomenologically adjusting the observed distribution by a polynomial function of $(w_o - w)$:

$$\frac{d\sigma}{dw} = \frac{1}{(w_o^2 - w^2)^2 + w_o^2\Gamma^2} w_o^2\Gamma_o^2 F_l^2(q)\mathbb{P}(w_o - w) \tag{100}$$

For example, BNL-E852 [**?**], has taken the following form in one of their analyses

$$\frac{d\sigma}{dw} = \frac{1}{(w_o^2 - w^2)^2 + w_o^2\Gamma^2} w_o^2\Gamma_o^2 F_l^2(q)[a_l + b_l(w - w_l^o) + c_l(w - w_l^o)^2]. \tag{101}$$

Fitting this distribution to a region with an identified mass distribution enhancement , we obtain values for w_o and Γ_o. However, as discussed earlier, a mass peak (enhancement) in a mass distribution is not akin of a resonance. A phase motion study must also be done and we come to this next.

2 Software Structure

2.1 Philosophy and General Outline

2.2 Preparing Data and Monte Carlo

2.3 Likelihood Fitting

2.4 Utilities and Graphic Tools

2.5 Using Xeon-Phi coprocessors

3 Example

3.1 PWA of $\gamma p \to pK^+K^-$ events in the ϕ mass range.

4 References

References

[1] C. W. Salgado and D. P. Weygand; Phys.Rept. 537 (2014) 1-58.

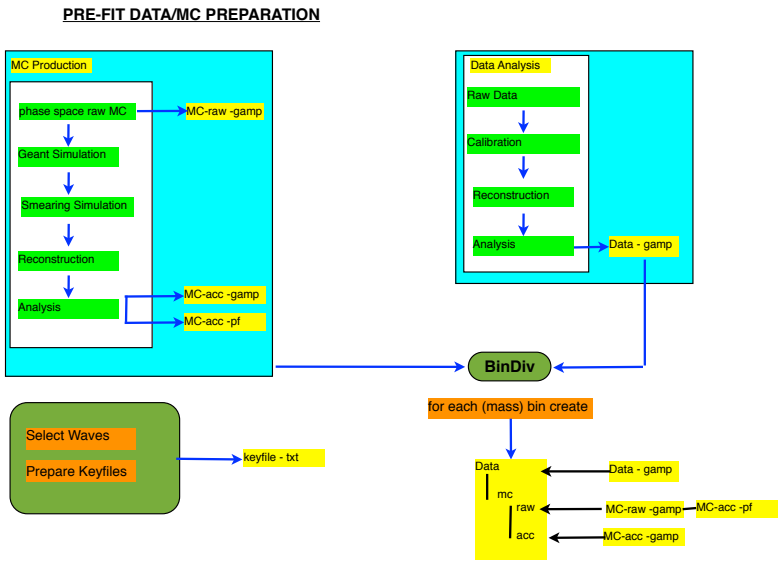


Figure 4: Data Preparation

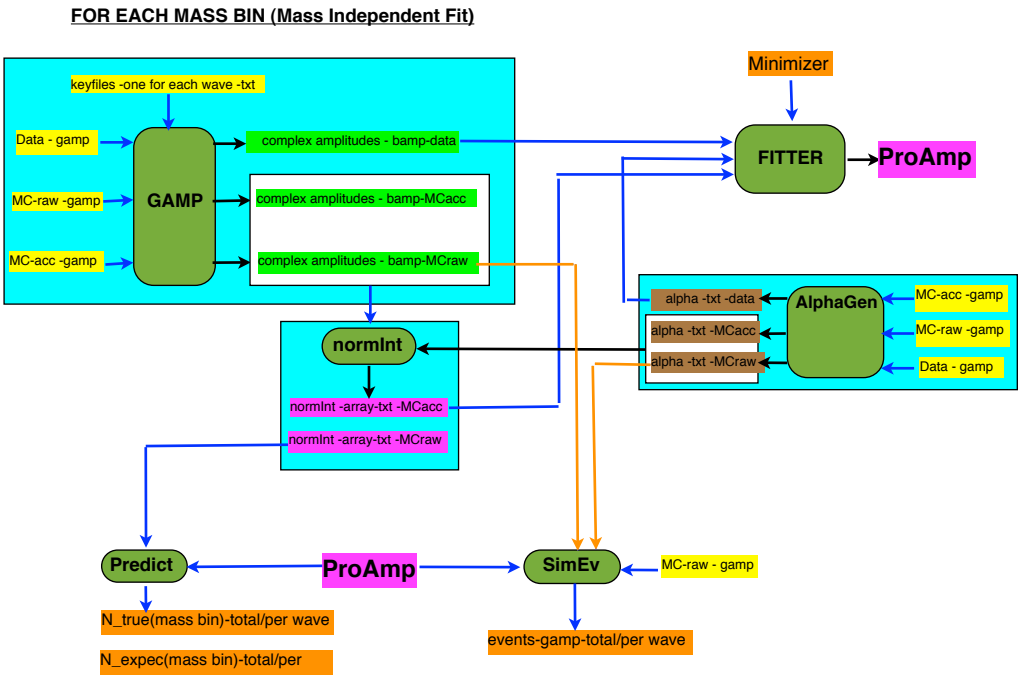


Figure 5: Fitting

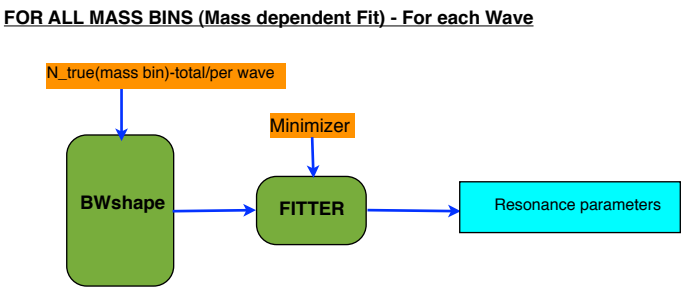


Figure 6: Mass Fitting

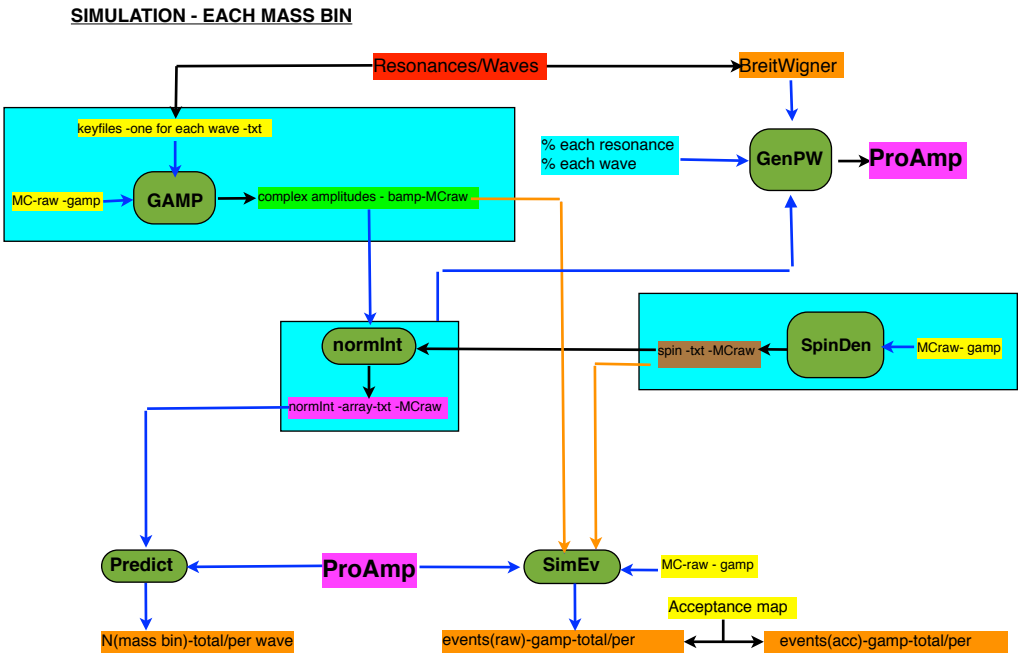


Figure 7: Simulation

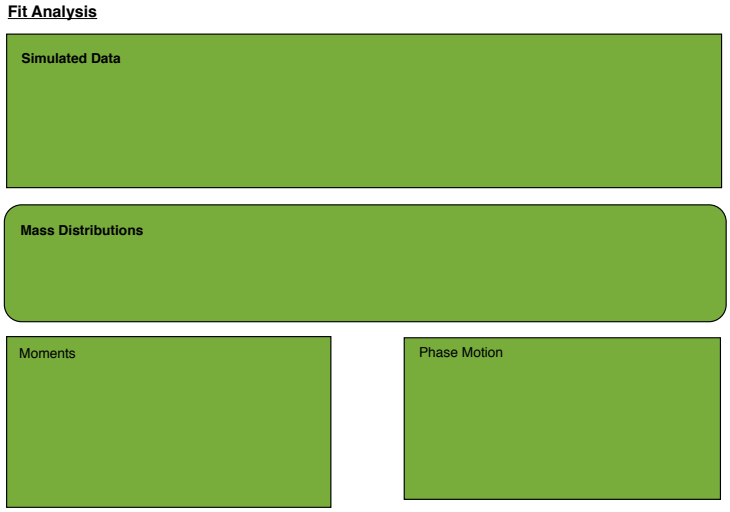


Figure 8: Simulation